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The Fibonacci sequence modulo π, chaos and some rational recursive equations

Mohamed Ben H. Rhouma

Department of Mathematics and Statistics, Sultan Qaboos University, PO Box 36, P.C. 123, Al-khod, Muscat, Oman

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Abstract

In this paper, we provide the closed form solution to the inter-related equations

$$y_{n+2} = \frac{y_n y_{n+1} - 1}{y_n + y_{n+1}} \quad \text{and} \quad z_{n+2} = (z_{n+1} + z_n) \mod \pi.$$

Both of these equations were suggested as open problems in the book by Kocic and Ladas [V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of High Order with Applications, Kluwer Academic, Dordrecht, 1993]. We also give the closed form solution to the equations

$$x_{n+1} = \frac{x_n x_{n-2} + a}{x_n + x_{n-2}} \quad \text{and} \quad x_{n+1} = \frac{x_{n-1} x_{n-2} + a}{x_{n-1} + x_{n-2}},$$

studied by X. Li and D. Zhu [X. Li, D. Zhu, Two rational recursive sequences, Comput. Math. Appl. 47 (2004) 1487–1494].

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E-mail address: rhouma@squ.edu.om.

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1. Introduction

Rational recursive sequences have been the focus of extensive research especially since the appearance of the monograph by Kocic and Ladas [4]. The more recent monograph by Kulenovic and Ladas [5] is exclusively dedicated to rational recursive equations and contains a long list of open problems and conjectures.

The following open problem was suggested in [4].

Open Problem [4, 6.11.1, p. 175]. The secant method for finding $\sqrt{-1}$: *Study the asymptotic behavior and the periodic nature of solutions of the difference equation*

$$y_{n+2} = \frac{y_n y_{n+1} - 1}{y_n + y_{n+1}}, \quad n = 0, 1, \dots, \tag{1}$$

where y_0 and y_1 are arbitrary numbers such that y_n exists for all n.

The change of variables $y_n = \cot z_n$, leads to the equation

$$\cot z_{n+2} = \frac{\cot z_n \cot z_{n+1} - 1}{\cot z_n + \cot z_{n+1}},$$

and so if the values of z_n are restricted to the interval $(0, \pi)$, we obtain the Fibonacci sequence modulo π, namely

$$z_{n+2} = (z_n + z_{n+1}) \mod \pi. \tag{2}$$

The study of this equation is also listed in [4] as:

Open Problem [4, 6.11.2, p. 175]. The Fibonacci sequence modulo π: *Assume z_0 and z_1 are arbitrary numbers in $(0, \pi)$. Study the asymptotic behavior and the periodic character of solutions of the difference equation* (2).

As explained in [4], a possible derivation of Eq. (1) comes from using Newton's method for solving the equation $f(x) = x^2 + 1 = 0$ with the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

and replacing the derivative $f'(x_n)$ by the approximation

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

Actually, the intersection between Fibonacci numbers and Newton's method is not restricted to this example. In fact, Gill and Miller [3] produced the subsequence $\{F_{2^n+1}/F_{2^n}\}$ using Newton's method to solve the equation $x^2 - x - 1 = 0$ with initial approximation $x_0 = 1$. In the same note [3], they also proved that the secant method with initial approximations $x_0 = 1$ and $x_1 = 2$ produces the sequence $\{F_{F_n+1}/F_{F_n}\}$. A more recent paper [9] also extended some old results [1] about $\sum(\pm 1)^n/(F_n F_{n+1})$, also using Newton's method.

Recently Li and Zhu [6] have studied similar looking equations, namely

$$x_{n+3}=\frac{a+x_{n+2}x_n}{x_{n+2}+x_n},\quad n=0,1,2,\ldots, \tag{3}$$

$$x_{n+3}=\frac{a+x_{n+1}x_n}{x_{n+1}+x_n},\quad n=0,1,2,\ldots, \tag{4}$$

where $a\in[0,\infty)$ and the initial conditions x_0, x_1, and x_2 are positive. For both Eqs. (3) and (4), it was found that the positive equilibrium of the equation is globally asymptotically stable.

In a recent paper [2], we have obtained closed form solutions for several equations, some of which having similar numerators and denominators to those of recursions (1), (3), and (4). In particular, the solutions to the following equations

$$x_{n+2}=-\frac{x_{n+1}x_n+1}{x_{n+1}+x_n}, \tag{5}$$

$$x_{n+2}=\frac{x_{n+1}x_n-1}{x_{n+1}-x_n}, \tag{6}$$

$$x_{n+2}=-\frac{x_{n+1}x_n-1}{x_{n+1}-x_n}, \tag{7}$$

$$x_{n+2}=-\frac{x_{n+1}x_n+x_n}{x_{n+1}-x_n}, \tag{8}$$

$$x_{n+2}=\frac{x_{n+1}-x_n}{x_{n+1}x_n+1}, \tag{9}$$

$$x_{n+2}=-\frac{x_{n+1}-x_n}{x_{n+1}x_n+1}, \tag{10}$$

as well as their *good* sets were given explicitly. For example, when

$$\left[\left(\frac{x_0-1}{x_0+1}\right)^{F_{n-2}}\left(\frac{x_1+1}{x_1-1}\right)^{F_{n-1}}\right]^{(-1)^n}\neq 1,\quad \forall n\geqslant 1,$$

and $(x_0-x_1)(x_0^2-1)(x_1^2-1)\neq 0$, solutions to (6) exist and are given by

$$x_n=\frac{1+\left[\left(\frac{x_0-1}{x_0+1}\right)^{F_{n-2}}\left(\frac{x_1+1}{x_1-1}\right)^{F_{n-1}}\right]^{(-1)^n}}{1-\left[\left(\frac{x_0-1}{x_0+1}\right)^{F_{n-2}}\left(\frac{x_1+1}{x_1-1}\right)^{F_{n-1}}\right]^{(-1)^n}}, \tag{11}$$

where F_n denote the nth term in the Fibonacci sequence. On the other hand, all solutions to (5) are periodic of prime period 3, while all solutions to (7) are periodic of prime period 6.

In this paper we provide:

(1) The closed form solutions to Eqs. (1), (3) and (4) as well as their asymptotic behavior.
(2) Results related to Eq. (2) from our discussion of the solution to Eq. (1). In particular, we establish that the Fibonacci sequence modulo π is chaotic, and give the exact number of periodic points of period p.
(3) A new set of solvable rational recursive equations.

The rest of the paper is organized as follows: in the next section we provide a few preliminary lemmas. In Section 3, we find and discuss the closed form solution of Eq. (1) along with results and comments about (2). In Section 4, we give the closed form solution to both (3) and (4) and explain why their solutions always converge to a fixed value. Finally, a few generalizations are made in Section 5.

2. Preliminary results

We begin with the following elementary lemma.

Lemma 1. *Let b, x, y, and z be arbitrary complex numbers such that*

$$b(x+y)(x+b)(y+b) \neq 0.$$

If

$$\frac{z-b}{z+b} = \left(\frac{x-b}{x+b}\right)\left(\frac{y-b}{y+b}\right), \tag{12}$$

then

$$z = \frac{xy+b^2}{x+y}. \tag{13}$$

The following lemmas give the solution to a few nonlinear sequences of the form

$$x_{n+1} = \alpha x_{n-k}^{p} x_{n-l}^{q}.$$

Lemma 2. *Let F_n be the Fibonacci sequence defined by*

$$F_0 = F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

The sequence $x_{n+2} = \alpha x_{n+1} x_n$, with initial conditions x_0 and x_1, has the general form

$$x_n = \alpha^{(-1+F_n)} x_0^{F_{n-2}} x_1^{F_{n-1}} \quad \text{for all integers } n \geqslant 2. \tag{14}$$

The proof of Lemma 2 can be easily obtained by induction.

Lemma 3. *Let λ_1, λ_2 and $\bar{\lambda}_2$ be the roots of the polynomial*

$$\lambda^3 - \lambda - 1 = 0, \tag{15}$$

and (a, b, c) be the solution to the linear system

$$\begin{aligned} &a + b + c = 0, \\ &a\lambda_1 + b\lambda_2 + c\bar{\lambda}_2 = 1, \\ &a\lambda_1^2 + b\lambda_2^2 + c\bar{\lambda}_2^2 = 0. \end{aligned} \tag{16}$$

Then, the sequence $x_{n+3} = x_{n+1}x_n$ with the initial conditions x_0, x_1, and x_2, has the general form

$$x_n = x_0^{A_{n-2}} x_1^{A_n} x_2^{A_{n-1}}, \tag{17}$$

where A_n is given by

$$A_n = a\lambda_1^n + b\lambda_2^n + c\bar{\lambda}_2^n. \tag{18}$$

Proof. It is easy to establish that given initial conditions x_0, x_1, and x_2, the general term of x_n can be written in the form

$$x_n = x_0^{C_n} x_1^{A_n} x_2^{B_n},$$

where A_n, B_n, C_n satisfy the linear recursion

$$Y_{n+3} = Y_{n+1} + Y_n. \tag{19}$$

Moreover, $C_0 = B_2 = A_1 = 1$, and $C_1 = C_2 = B_0 = B_1 = A_0 = A_2 = 0$. The values of λ_1, λ_2, and $\bar{\lambda}_2$ are the roots the polynomial $\lambda^3 - \lambda - 1 = 0$ which is the characteristic polynomial of (19). The rest of the proof follows from straightforward computations. $\square$

Lemma 4. *Let μ_1, μ_2, and $\bar{\mu}_2$ be the roots of the polynomial*

$$\mu^3 - \mu^2 - 1 = 0, \tag{20}$$

and let (a, b, c) be the solution to the linear system

$$\begin{aligned} &a + b + c = 0,\\ &a\mu_1 + b\mu_2 + c\bar{\mu}_2 = 0,\\ &a\mu_1^2 + b\mu_2^2 + c\bar{\mu}_2^2 = 1. \end{aligned} \tag{21}$$

Then, the sequence $x_{n+3} = x_{n+2}x_n$ with the initial conditions x_0, x_1, and x_2, has the general form

$$x_n = x_0^{B_{n-1}} x_1^{B_{n-2}} x_2^{B_n}, \tag{22}$$

where B_n is given by

$$B_n = a\mu_1^n + b\mu_2^n + c\bar{\mu}_2^n. \tag{23}$$

The proof of Lemma 4 is identical to that of Lemma 3 and thus will be omitted.

3. Equation (1) and the Fibonacci sequence mod π

First, notice that by Lemma 1, a solution to (1) also satisfies the equation

$$\frac{y_{n+2} - i}{y_{n+2} + i} = \left(\frac{y_{n+1} - i}{y_{n+1} + i}\right)\left(\frac{y_n - i}{y_n + i}\right). \tag{24}$$

By virtue of Lemma 2, we obtain that

$$\frac{y_n - i}{y_n + i} = \left(\frac{y_0 - i}{y_0 + i}\right)^{F_{n-2}} \left(\frac{y_1 - i}{y_1 + i}\right)^{F_{n-1}}.$$

Thus setting up

$$e^{i\theta_0} = \frac{y_0 - i}{y_0 + i} \quad \text{and} \quad e^{i\theta_1} = \frac{y_1 - i}{y_1 + i}, \tag{25}$$

or equivalently

$$\theta_0 = -2 \operatorname{arccot} y_0 \quad \text{and} \quad \theta_1 = -2 \operatorname{arccot} y_1, \tag{26}$$

we have

$$\frac{y_n - i}{y_n + i} = e^{i(F_{n-2}\theta_0 + F_{n-1}\theta_1)},$$

and

$$y_n = i\,\frac{1 + e^{i(F_{n-2}\theta_0 + F_{n-1}\theta_1)}}{1 - e^{i(F_{n-2}\theta_0 + F_{n-1}\theta_1)}}. \tag{27}$$

Theorem 1. *The solution to Eq.* (1) *exists for all integers* $n \geqslant 0$ *if and only if*

$$(F_{n-2}\theta_0 + F_{n-1}\theta_1) \mod 2\pi \neq 0 \quad \textit{for all } n \geqslant 0. \tag{28}$$

Moreover, when it exists the solution to (1) *is given by*

$$\begin{aligned} y_n &= -\cot\left(\frac{F_{n-2}\theta_0 + F_{n-1}\theta_1}{2}\right) \\ &= \cot(F_{n-2}\operatorname{arccot} y_0 + F_{n-1}\operatorname{arccot} y_1). \end{aligned} \tag{29}$$

In addition,

(i) *If* θ_0 *and* θ_1 *are both rational multiples of* π, *then either* $\{y_n\}$ *diverges in finitely many steps or* y_n *is periodic.*
(ii) *If* θ_0 *is a rational multiple of* π *and* θ_1 *is not* (*or vice-versa*), *then* $\{y_n\}$ *is aperiodic and does exist for all integers.*

Proof. Both (28) and (29) follow straight from (27). On the other hand, if $\theta_0 = (m_0\pi/n_0)$ and $\theta_1 = (m_1\pi/n_1)$, then studying the sequence $(F_{n-2}\theta_0 + F_{n-1}\theta_1) \bmod 2\pi$ is equivalent to studying the sequence

$$G_n = (m_0 n_1 F_{n-2} + m_1 n_0 F_{n-1}) \mod 2n_0 n_1, \tag{30}$$

which is bound to be periodic. (See, for example, [7] and references therein.) If G_n contains zero then the sequence $\{y_n\}$ will diverge. Otherwise, the sequence $\{y_n\}$ will be periodic. □

To illustrate item (i) of Theorem 1, the initial values

$$y_0 = 1 + \sqrt{2} \quad \text{and} \quad y_1 = 0,$$

do correspond to values of θ_0 and θ_1 that are rational multiples of π, and yield a periodic sequence of prime period 12. On the other hand, when

$$y_0 = 0 \quad \text{and} \quad y_1 = \sqrt{3},$$

the sequence $\{y_n\}$ only exist for $n \leqslant 7$.

Consider now T_π the square with corners $(0,0)$, $(0,\pi)$, (π,π), and $(\pi,0)$ in the plane and the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{31}$$

Using the definitions and results of [8, Chapter 5], we have that the matrix A induces a hyperbolic Anosov toral automorphism defined on T_π, namely the map

$$L_A : T_\pi \to T_\pi \quad \text{defined by } L_A(x,y) = (y, x+y \bmod \pi). \tag{32}$$

The following relation between the nth power of the matrix A and the Fibonacci sequence is well known:

$$A^n = \begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix}.$$

Also known is the fact that the matrix A^2 induces the *cat map*, one of the simplest examples of chaos in two dimensions. Perhaps less known, is that the map L_A induced by the matrix A is itself chaotic. The following theorem is just a translation of the fact that L_A is chaotic in terms of the Fibonacci sequence mod π.

Theorem 2.

(i) *The set of periodic points of the Fibonacci sequence modulo π is dense in T_π. Moreover, for any given integer $n \geqslant 1$ the number of n-periodic points on the Torus T_π is given by*

$$F_{n-2} + F_n - 1 + (-1)^{n-1}. \tag{33}$$

(ii) *Given any two open neighborhoods U and V in T_π, there exist initial conditions $(z_0, z_1) \in U$ and an integer $n \geqslant 0$ such that the nth iterate (z_n, z_{n+1}) of the Fibonacci sequence modulo π belongs to the set V.*

Proof. Items (i) and (ii) are simply the density of periodic points and the transitivity properties of chaotic systems. As for the number of periodic solutions of period n, it can be obtained by solving the equation

$$A^n (z_0, z_1)^T = (z_0, z_1)^T \mod \pi,$$

and using the identity

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^n. \qquad \square$$

In terms of the sequence (1), Theorem 2 implies that if $\mathcal{F}$ and $\mathcal{P}$ are respectively the forbidden set, and the set of initial conditions of periodic solutions of (1), then $\mathcal{F} \cup \mathcal{P}$ is dense in $\mathbb{R}^2$. Theorem 2 also implies transitivity of the sequence (1) in $\mathbb{R}^2$.

4. Equations of the form $x_{n+1} = (x_{n-k}x_{n-l} + 1)/(x_{n-k} + x_{n-l})$

When $a > 0$ and k and l are distinct nonnegative integers, all recursions of the form

$$y_{n+2} = \frac{y_{n-k+1}y_{n-l+1} + a}{y_{n-k+1} + y_{n-l+1}}, \tag{34}$$

can be transformed to the equation

$$x_{n+2} = \frac{x_{n-k+1}x_{n-l+1} + 1}{x_{n-k+1} + x_{n-l+1}}, \tag{35}$$

by the change of variables $y_n = \sqrt{a}x_n$.

In this section we will present results for the case $k = 0$ and $l = 1$. With minor changes all results do apply to other cases. In particular, Eqs. (3) and (4) studied in [6] can be treated in a very similar way.

Consider the equation

$$x_{n+2} = \frac{x_n x_{n+1} + 1}{x_n + x_{n+1}}. \tag{36}$$

By virtues of Lemmas 1 and 2, we have that

$$\left(\frac{x_n - 1}{x_n + 1}\right) = \left(\frac{x_0 - 1}{x_0 + 1}\right)^{F_{n-2}} \left(\frac{x_1 - 1}{x_1 + 1}\right)^{F_{n-1}}. \tag{37}$$

Thus the expression of x_n is easily obtainable whenever

$$(x_0 - 1)^{F_{n-2}}(x_1 - 1)^{F_{n-1}} \neq (x_0 + 1)^{F_{n-2}}(x_1 + 1)^{F_{n-1}},$$

to yield the closed form

$$x_n = \frac{(x_0 + 1)^{F_{n-2}}(x_1 + 1)^{F_{n-1}} + (x_0 - 1)^{F_{n-2}}(x_1 - 1)^{F_{n-1}}}{(x_0 + 1)^{F_{n-2}}(x_1 + 1)^{F_{n-1}} - (x_0 - 1)^{F_{n-2}}(x_1 - 1)^{F_{n-1}}}. \tag{38}$$

Notice also that using Eq. (37), we can write

$$\left|\frac{x_n - 1}{x_n + 1}\right| = \left|\frac{x_0 - 1}{x_0 + 1}\right|^{F_{n-2}} \left[\left|\frac{x_1 - 1}{x_1 + 1}\right|^{\frac{F_{n-1}}{F_{n-2}}}\right]^{F_{n-2}}. \tag{39}$$

Since the ratio $F_{n-1}/F_{n-2} \to (1 + \sqrt{5})/2$ as $n \to \infty$, we can conclude that if

$$\left|\frac{x_0 - 1}{x_0 + 1}\right| \left|\frac{x_1 - 1}{x_1 + 1}\right|^{\frac{1+\sqrt{5}}{2}} > 1,$$

then the sequence $\{x_n\} \to -1$. Similarly, if

$$\left|\frac{x_0 - 1}{x_0 + 1}\right| \left|\frac{x_1 - 1}{x_1 + 1}\right|^{\frac{1+\sqrt{5}}{2}} < 1,$$

then the sequence $\{x_n\} \to 1$ as $n \to \infty$.

An immediate consequence of this fact is that if x_0 and x_1 are both positive then the sequence $\{x_n\} \to 1$ as $n \to \infty$. A very similar explanation works for Eqs. (3) and (4). In fact, the solutions to Eqs. (3) and (4) with $a = 1$ are respectively given by

$$x_n = \frac{1 + \left(\frac{x_0-1}{x_0+1}\right)^{A_{n-2}} \left(\frac{x_1-1}{x_1+1}\right)^{A_n} \left(\frac{x_2-1}{x_2+1}\right)^{A_{n-1}}}{1 - \left(\frac{x_0-1}{x_0+1}\right)^{A_{n-2}} \left(\frac{x_1-1}{x_1+1}\right)^{A_n} \left(\frac{x_2-1}{x_2+1}\right)^{A_{n-1}}} \tag{40}$$

and

$$x_n = \frac{1 + \left(\frac{x_0-1}{x_0+1}\right)^{B_{n-1}} \left(\frac{x_1-1}{x_1+1}\right)^{B_{n-2}} \left(\frac{x_2-1}{x_2+1}\right)^{B_n}}{1 - \left(\frac{x_0-1}{x_0+1}\right)^{B_{n-1}} \left(\frac{x_1-1}{x_1+1}\right)^{B_{n-2}} \left(\frac{x_2-1}{x_2+1}\right)^{B_n}}, \tag{41}$$

where A_n and B_n are respectively given by Lemmas 3 and 4. Since Li and Zhu [6] considered positive initial conditions and that both sequences $\{A_n\}$ and $\{B_n\}$ converge to $+\infty$ as $n \to \infty$ we can easily see that $\{x_n\} \to 1$ for both Eqs. (3) and (4). In fact, in light of Eqs. (40) and (41), it is also easy to see that the sequence $\{x_n\}$ will converge to -1 for all initial conditions $x_0, x_1, x_2 \in (-\infty, 0)$. Moreover, obtaining the good sets of (40) and (41) and more detailed rules of convergence, is done in a very similar way as for Eq. (36).

The following theorem sums up the discussion of this section.

Theorem 3. *Let $k > l > 0$ be two integers and consider Eq.* (35) *with initial conditions $x_0, x_1, \ldots, x_k$. There exist sequences $\{A_n^{(i)}\}$ for $i = 0, \ldots, k$ all converging to ∞ as $n \to \infty$ such that*

$$x_n = \frac{\prod_{i=0}^{k} (x_i + 1)^{A_n^{(i)}} + \prod_{i=0}^{k} (x_i - 1)^{A_n^{(i)}}}{\prod_{i=0}^{k} (x_i + 1)^{A_n^{(i)}} - \prod_{i=0}^{k} (x_i - 1)^{A_n^{(i)}}}, \tag{42}$$

as long as the denominator in (42) *does not vanish. Moreover, if all initial conditions are positive then $\{x_n\} \to 1$ and if all initial conditions are negative then $\{x_n\} \to -1$.*

5. Generalizations to other rational recursive equations

In this section, we illustrate how to obtain closed form solutions for some special rational recursive equations. To this purpose, consider the equation

$$x_{n+k+1} = \alpha \prod_{i=0}^{k} x_{n+i}^{p_i}, \quad n = 0, 1, 2, \ldots \tag{43}$$

with initial conditions $x_0, x_1, \ldots, x_k$. For simplicity we will consider only integer powers p_i. As in Lemmas 2–4, by computing the first few items of the sequence $\{x_n\}$, we can easily obtain the general form of the solution, namely

$$x_n = \alpha^{B_n} \prod_{i=0}^{k} x_i^{A_n^{(i)}}, \tag{44}$$

where the sequences $\{A_n^{(i)}\}$ satisfy the linear equation

$$A_{n+k+1}^{(i)} = \sum_{j=0}^{k} p_j A_{n+j}^{(i)} \tag{45}$$

with initial conditions $A_i^{(i)} = 1$ and $A_j^{(i)} = 0$ for all $0 \leqslant i \neq j \leqslant k$. The sequence $\{B_n\}$ also satisfies the linear equation

$$B_{n+k+1} = 1 + \sum_{j=0}^{k} p_j B_{n+j}, \qquad B_0 = B_1 = \cdots = B_k = 0. \tag{46}$$

Appropriate changes of variables do lead to several interesting cases, of which we only choose the following few.

5.1. A first order equation

We start with the elementary first order equation

$$x_{n+1} = \alpha x_n^2 \tag{47}$$

with initial condition x_0. It is easy to establish that the general term of x_n is given by

$$x_n = \alpha^{2^n - 1} x_0^{(2^n)}. \tag{48}$$

Immediately we can see that the closed form of the first order two-parameter recursive equation

$$y_{n+1} = \alpha y_n^2 + 2\alpha a y_n + a(a\alpha - 1), \tag{49}$$

is given by

$$y_n = \alpha^{2^n - 1}(y_0 + a)^{(2^n)} - a. \tag{50}$$

In particular, when $\alpha a = 1$ we obtain that the closed form to

$$y_{n+1} = \alpha y_n^2 + 2y_n$$

is given by

$$y_n = \alpha^{2^n - 1}\left(y_0 + \frac{1}{\alpha}\right)^{(2^n)} - \frac{1}{\alpha}.$$

With the change of variables $x_n = (y_n - a)/(y_n + a)$, we can also deduce from (50) the closed form solutions to the 2-parameter equation

$$y_{n+1} = \frac{(1+\alpha)a y_n^2 + 2(1-\alpha)a^2 y_n + (1+\alpha)a^3}{(1-\alpha)y_n^2 + 2(1+\alpha)a y_n + (1-\alpha)a^2} \tag{51}$$

and its special cases when $\alpha = \pm 1$

$$y_{n+1} = \frac{y_n^2 + a^2}{2y_n} \tag{52}$$

$$y_{n+1} = \frac{2a^2 y_n}{y_n^2 + a^2}. \tag{53}$$

In fact, the solution to (51) is given by

$$y_n = a\frac{(y_0+a)^{(2^n)} + \alpha^{2^n-1}(y_0-a)^{(2^n)}}{(y_0+a)^{(2^n)} - \alpha^{2^n-1}(y_0-a)^{(2^n)}}, \tag{54}$$

whenever the denominator in (54) does not vanish for all $n \geqslant 0$.

5.2. Some high order equations

Let $l > 1$ be an integer, a be an arbitrary real number, and $x_0 \neq -a$, $x_1 \neq -a, \ldots,$ $x_l \neq -a$ be the initial conditions of the rational recursive equation

$$x_{n+l+1} = a\frac{x_{n+l} - x_n}{x_n + a}, \quad n = 0, 1, 2, \ldots. \tag{55}$$

Using the change of variables $x_n = a(y_n - 1)$, Eq. (55) reduces to the simpler form

$$y_{n+l+1} = \frac{y_{n+l}}{y_n}, \quad n = 0, 1, 2, \ldots,$$

which is well defined whenever the product

$$\prod_{i=0}^{l} y_i \neq 0.$$

Our first conclusion then is that Eq. (55) is well defined if and only if

$$\prod_{i=0}^{l} (x_i + a) \neq 0.$$

Second, whenever the polynomial $\lambda^{l+1} - \lambda^l + 1$ is solvable, Eq. (55) is also solvable explicitly. For the sake of this discussion we will just concentrate on the general behavior of Eq. (55) for large values of l. Using Corollary 1.2.1, p. 6, and Theorem 1.3.6, p. 12, in [4], we can easily establish that unless $x_0 = x_1 = \cdots = x_l = 0$, as n grows large all solutions to Eq. (55) will alternate between terms converging to zero and others diverging to ∞.

The second and last equation we consider is the rational recursive equation

$$x_{n+2} = \frac{x_{n+1}x_n}{x_{n+1} + x_n + 1}, \quad n = 0, 1, 2, \ldots, \tag{56}$$

with initial conditions x_0 and x_1. Setting $y_n = x_n/(x_n + 1)$, we obtain that

$$y_{n+2} = y_{n+1}y_n.$$

Thus, by Lemma 2 and some simple algebra, we obtain that

$$x_n = \frac{x_0^{F_{n-2}} x_1^{F_{n-1}}}{(1+x_0)^{F_{n-2}}(1+x_1)^{F_{n-1}} - x_0^{F_{n-2}} x_1^{F_{n-1}}}, \tag{57}$$

whenever the initial conditions x_0 and x_1 are chosen such that the denominator never vanishes.

We would like to end this paper by stressing that despite the fact that we can generate several closed form solutions for some interesting rational recursive equations, the “set” of solvable equations remains very small. In fact, to our knowledge, amongst the nontrivial equations of the form

$$x_{n+2} = \frac{\alpha x_{n+1} + \beta x_n + \gamma}{a x_{n+1} + b x_n + c}$$

only the following special equations have closed form solutions:

$$\begin{aligned} x_{n+2} &= \frac{\alpha x_{n+1}}{x_n}, \\ x_{n+2} &= \frac{\alpha x_n}{x_{n+1}}, \\ x_{n+2} &= \frac{\alpha x_{n+1} - \beta x_n + \alpha\beta - \beta^2}{x_n + \beta}, \\ x_{n+2} &= \frac{\alpha x_n - \beta x_{n+1} + \alpha\beta - \beta^2}{x_{n+1} + \beta}. \end{aligned} \tag{58}$$

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